

CARTER'S CONSTANT REVEALED

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A new formulation of Carter's constant for geodesic motion in Kerr black holes is given. It is shown that Carter's constant corresponds to the total angular momentum plus a precisely defined part which is quadratic in the linear momenta. The characterization is exact in the weak field limit obtained by letting the gravitational constant go to zero. It is suggested that the new form can be useful in current studies of the dynamics of extreme mass ratio inspiral (EMRI) systems emitting gravitational radiation.

PACS numbers: 04.70.-s, 04.25.-g

1 INTRODUCTION

The Kerr solution together with its charged counterpart represents one of the most important advances in general relativity since its original formulation by Einstein almost a century ago. As a stationary axisymmetric geometry it admits one timelike and one spacelike Killing vector field. The geodesic motion therefore has two constants of the motion which are linear in the momenta corresponding to conservation of energy and conservation of the azimuthal angular momentum. As always, there is also the quadratic constant of the motion corresponding to conservation of mass.

Carter found an additional nontrivial quadratic constant of the motion [1]. Ever since its discovery almost four decades ago, there has been an ongoing discussion about the physical interpretation of Carter's constant, see e.g. [2, 3, 4, 5]. The constant is important when calculating the spectrum of gravitational waves emanating from the dynamics of extreme mass ratio inspiral (EMRI) type systems in particular (see [6] for a review). The consensus in the existing literature has been that Carter's constant represents a generalization of the total squared angular momentum. Indeed, in the limit of zero rotation $a \rightarrow 0$ it does reduce to the total angular momentum (minus the conserved azimuthal part). However, there has also been a consensus that it is not exactly the total angular momentum, but that it corresponds to the total angular momentum plus an additional part. In this note, we give a precise and unambiguous characterization of Carter's constant. In addition to the zero angular momentum limit, $a \rightarrow 0$, there is also another limit which is physically relevant. This is the weak field limit $G \rightarrow 0$ in which Newton's gravitational constant tends to zero. For example, the azimuthal angular momentum L_z is the same in both these limits. However, it turns out that Carter's constant does not reduce to a pure angular momentum in the weak field limit. Instead, there are additional terms quadratic in the linear momenta which are proportional to the square of a , the angular momentum parameter in the Kerr geometry. Below, we give a more detailed description and interpretation of this result.

The limit $G \rightarrow 0$ is mathematically equivalent to the limit $M \rightarrow 0$ corresponding to the source mass tending to zero since G only occurs in the the combination GM in the Kerr solution. From the physical point of view however, the limit $M \rightarrow 0$ (while keeping a fixed) can only be taken, strictly speaking, for overextreme systems ($M < a$), which hence do not include black holes. However, it is reasonable to expect that the physical interpretation of Carter's constant should in principle be the same for both underextreme and overextreme systems. Mathematically, the limit $M \rightarrow 0$ is well defined for any Kerr solution. Besides, overextreme astrophysical systems are interesting in themselves as they are presumably not uncommon. Notably, they include the solar system for which $a/M \approx 40$ [7] where $a = J/M$ and J is the total angular momentum. It also worth noting that for charged black holes, the limits $G \rightarrow 0$ and $M \rightarrow 0$ are not equivalent.

2 THE IDEA

In order to formulate our basic idea, it is useful to express the Kerr metric in terms of a suitable Minkowski frame M^μ meaning that $\eta = \eta_{\mu\nu} M^\mu M^\nu$ is the Minkowski metric. We use relativistic units with $c = 1$, but keep G explicit. Then we define a Minkowski frame expressed in Boyer-Lindquist coordinates by

$$M^0 = \frac{\rho_0}{\rho} (dt - a \sin^2 \theta d\phi) , \quad M^1 = \frac{\rho}{\rho_0} dr , \quad M^2 = \rho d\theta , \quad M^3 = \frac{\sin \theta}{\rho} (-adt + \rho_0^2 d\phi) \quad (1)$$

where the functions ρ and ρ_0 are given by

$$\rho^2 = r^2 + a^2 \cos^2 \theta , \quad \rho_0^2 = r^2 + a^2 . \quad (2)$$

This gives the Minkowski metric in oblate spheroidal coordinates. They are related to Cartesian coordinates by

$$\begin{aligned} x &= \rho_0 \sin \theta \cos \phi \\ y &= \rho_0 \sin \theta \sin \phi \\ z &= r \cos \theta . \end{aligned} \quad (3)$$

The explicit form of the Minkowski metric in these coordinates is

$$\eta = \tilde{\eta}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = -dt^2 + \frac{\rho^2}{\rho_0^2} dr^2 + \rho^2 d\theta^2 + \rho_0^2 \sin^2 \theta d\phi^2 , \quad (4)$$

where $\tilde{x}^\mu = (t, r, \theta, \phi)$ denotes the spheroidal coordinates. The Kerr metric itself is

$$g = -h(M^0)^2 + h^{-1}(M^1)^2 + (M^2)^2 + (M^3)^2 , \quad (5)$$

where

$$h = 1 - \frac{2GM r}{r^2 + a^2} . \quad (6)$$

Note that the Minkowski frame M^μ does not contain the mass¹ which appears only in the function h . In addition, the function h depends on only one of the coordinates, r . Taking the weak field limit $G \rightarrow 0$ of the Kerr metric (5) obviously gives back Minkowski spacetime in oblate spheroidal coordinates. Carter's constant is given by [8]

$$Q = p_\theta^2 + p_\phi^2 \cot^2 \theta - a^2 (p_t^2 + 2\mathcal{H}) \cos^2 \theta \quad (7)$$

where the Hamiltonian \mathcal{H} is related to the mass μ of the particle by

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = -\frac{1}{2} \mu^2 . \quad (8)$$

Using this relation and writing $E = -p_t$, $L_z = p_\phi$ for the energy and azimuthal angular momentum, the constant can be rewritten as

$$Q = p_\theta^2 + L_z^2 \cot^2 \theta - a^2 (E^2 - \mu^2) \cos^2 \theta . \quad (9)$$

In the limit of zero angular momentum of the source, $a \rightarrow 0$, we have as is well-known

$$Q = p_\theta^2 + L_z^2 \cot^2 \theta = L_x^2 + L_y^2 = L^2 - L_z^2 \quad (10)$$

¹The parameter a has the interpretation of angular momentum per unit mass. Since the angular momentum itself scales linearly with the mass, a should be considered as a parameter which does not depend on the mass.

where

$$L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \quad (11)$$

is the total squared angular momentum expressed in spherical coordinates and L_x and L_y are components of the angular momentum corresponding to equatorial symmetry axes.

In order to extend this result, we make the crucial observation that Carter's constant as given in (9) does not contain the source mass M explicitly. It contains only the parameters L_z , E and μ which refer to the particle. Therefore the expression in (9) must itself be a constant of the motion in flat space. At this point it is essential that oblate spheroidal coordinates are used in (9). We can now reinterpret the expression for Q in (9) using flat space spheroidal coordinates by setting $E = -p_t$ and $\mu^2 = -2\mathcal{H}_{\text{flat}} = -\tilde{\eta}^{\mu\nu}p_\mu p_\nu$ where $\tilde{\eta}^{\mu\nu}$ are the contravariant components of the metric (4). This gives

$$Q_0 = \frac{r^2}{\rho^2} p_\theta^2 + \frac{r^2 \cot^2\theta}{\rho_0^2} p_\phi^2 - \frac{a^2 \rho_0^2 \cos^2\theta}{\rho^2} p_r^2. \quad (12)$$

Where the subscript "0" is a reminder that this expression is valid in flat space only. It must necessarily correspond to some combination of the flat space constants of the motion. Guided by (10), we first try with $L_x^2 + L_y^2$. To express that constant in spheroidal coordinates we note that the total angular momentum is

$$L^2 = \frac{\rho_0^2}{\rho^4} (rp_\theta - a^2 \sin\theta \cos\theta p_r)^2 + \left(1 - \frac{a^2 \cos^2\theta}{\rho_0^2}\right) \frac{p_\phi^2}{\sin^2\theta}. \quad (13)$$

In the limit $a \rightarrow 0$, this reduces to (11) as it should. Subtracting $L_z^2 = p_\phi^2$ changes only the second term and we obtain

$$L_x^2 + L_y^2 = L^2 - L_z^2 = \frac{\rho_0^2}{\rho^4} (rp_\theta - a^2 \sin\theta \cos\theta p_r)^2 + \frac{r^2 \cot^2\theta}{\rho_0^2} p_\phi^2. \quad (14)$$

The flat space version of Carter's constant in (12) has no cross terms in the momenta whereas the above expression for $L_x^2 + L_y^2$ contains a cross term in $p_r p_\theta$. Therefore we need to add some other (flat space) constant of the motion which can cancel this cross term. It turns out that p_z^2 is exactly what is needed to do that job. Expressing p_z in oblate spheroidal coordinates gives

$$p_z = \frac{\rho_0^2 \cos\theta}{\rho^2} p_r - \frac{r \sin\theta}{\rho^2} p_\theta. \quad (15)$$

Finally combining the expressions (12), (14) and (15) gives the amazingly simple relation

$$Q_0 = L_x^2 + L_y^2 - a^2 p_z^2. \quad (16)$$

This is our main result. For future reference we note that the (flat space) total angular momentum can be written in the form

$$L^2 = Q_0 + L_z^2 + a^2 p_z^2. \quad (17)$$

We also define the related positive definite constant of the motion

$$P = Q + a^2 E^2 = p_\theta^2 + L_z^2 \cot^2\theta + a^2 E^2 \sin^2\theta + a^2 \mu^2 \cos^2\theta. \quad (18)$$

Its flat space form is

$$P_0 = L_x^2 + L_y^2 + a^2 (p_x^2 + p_y^2) + a^2 \mu^2. \quad (19)$$

3 INTERPRETATION

In this section we make some comments on the new characterization of Carter's constant given above. We begin by focusing attention on the form (16). It is then clear that the equatorial projection of the angular momentum and the momentum in the z -direction have a tendency to follow each other. This can be regarded as natural from the physical point of view, since both quantities are related to motion off the equatorial plane. Looking instead at the form (19) shows that the equatorially projected angular momentum has a tendency to oppose horizontal motion and vice versa. There is therefore a certain jet-like effect in that horizontal motion is disfavored unless it occurs in the equatorial plane itself. Based on the results discussed above, in particular

equation (17), one may define a total (non-conserved) angular momentum for a particle in the Kerr geometry by

$$\tilde{L}^2 := Q + L_z^2 + a^2 p_z^2, \quad (20)$$

where p_z is given by (15) and the momenta are defined with the Kerr metric factor h included, $p_r = h^{-1} \rho_0^{-2} \dot{\rho}^2$, $p_\theta = \rho^2 \dot{\theta}$. Referring to (15), the explicit form of the Kerr p_z momentum is therefore given by

$$p_z = h^{-1} \cos \theta \dot{r} - r \sin \theta \dot{\theta}. \quad (21)$$

It is clear that \tilde{L}^2 reduces to the conserved total angular momentum in both limits, $a \rightarrow 0$ and $G \rightarrow 0$. From its definition it is also evident that \tilde{L}^2 differs from the flat space angular momentum only by factors of h . We therefore expect that \tilde{L}^2 is approximately constant in the region $\tilde{h} \ll 1$ where $\tilde{h} = 1 - h = 2GMr/(r^2 + a^2)$. Since the horizon is defined by $h = 0$, this corresponds to the region well outside the horizon, $r \gg r_h$.

Glampedakis et al. [9] define a “spherical” total angular momentum by

$$\tilde{L}_{\text{GHK}}^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}. \quad (22)$$

Although formally identical to the expression (11), it should be borne in mind that (22) is an expression in spheroidal coordinates, not spherical. Therefore, although the Glampedakis et al. angular momentum function \tilde{L}_{GHK}^2 does reduce to the conserved angular momentum in the limit $a \rightarrow 0$, it does not in the limit $G \rightarrow 0$, for which the correct expression should be (13).

Our point of view is that since angular momenta should be related to (spatial) rotational symmetries, the only angular momentum which can be defined for the Kerr metric is $L_z = p_\phi$. Only by taking appropriate limits (such as $a \rightarrow 0$ and $G \rightarrow 0$) can one achieve a setting in which also other angular momenta can be defined. Comparing with the work of de Felice and Preti [3], they regard the spherical-like expression (22) as an angular momentum in their attempts to give a physical meaning to Carter’s constant. However, referring to the above discussion, such an interpretation does not have an invariant meaning. This is most certainly the reason why their additional term depends on p_r rather than p_z as obtained in our analysis.

Finally we may speculate that the insights gained in this work may prove practically useful in e.g. improving approximate methods for calculating gravitational waveforms from EMRIs, see e.g. [6]. In particular it would be interesting to check whether our \tilde{L}^2 is even more close to being conserved than \tilde{L}_{GHK}^2 .

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